



TITLE:

次元論における局所化について(集合論的位相と幾何学的位相)

AUTHOR(S):

YOKOI, KATSUYA

CITATION:

YOKOI, KATSUYA. 次元論における局所化について(集合論的位相と幾何学的位相). 数理解析研究所講究録 1995, 901: 58-61

ISSUE DATE:

1995-03

URL:

<http://hdl.handle.net/2433/59371>

RIGHT:

次元論における局所化について

横井 勝弥 (KATSUYA YOKOI)

筑波大学数学系

1. Introduction. Dennis Sullivan [S] pointed out the availability and applicability of localization methods in homotopy theory. We shall apply the methods to dimension theory.

Throughout this paper, we shall denote by \mathcal{P} the set of all prime numbers. The full subcategory of the category \mathcal{G} of all groups consisting of all nilpotent groups is denoted by \mathcal{N} . Let $P \subseteq \mathcal{P}$. A homomorphism $e: G \rightarrow G_P$ in \mathcal{N} is said to be a *P-localizing map* if G_P is *P-local* (i.e., $x \mapsto x^n$, $x \in G_P$, is bijective for all $n \in P'$, where $n \in P'$ means that n is a product of primes in the complementary collection P' of primes with respect to P) and if $e^*: \text{Hom}(G_P, K) \approx \text{Hom}(G, K)$ provided $K \in \mathcal{N}$, with K *P-local*. We know that there exists a *P-localization theory* on the category \mathcal{N} [H-M-R].

Definition. A connected CW-complex X is *nilpotent* if $\pi_1(X)$ is nilpotent group and operates nilpotently on $\pi_n(X)$ for every $n \geq 2$.

Let \mathcal{NH} be the homotopy category of nilpotent CW-complexes. \mathcal{NH} contains the homotopy category of simply connected CW-complexes. Moreover, the *simple* CW-complexes are plainly in \mathcal{NH} ; in particular, \mathcal{NH} contains all connected Hopf spaces.

Definition. Let $X \in \mathcal{NH}$ and $P \subseteq \mathcal{P}$. Then X is *P-local* if $\pi_n(X)$ is *P-local* for all $n \geq 1$. A map $f: X \rightarrow Y$ in \mathcal{NH} *P-localizes* if Y is *P-local* and

$$f^*: [Y, Z]_* \approx [X, Z]_*$$

for all *P-local* Z in \mathcal{NH} , where $[A, B]_*$ means the set of pointed homotopy classes of maps from A to B .

The following results ([Su], [H-M-R]) are very useful in this paper.

Theorem A. *Every X in \mathcal{NH} admits a P -localization.*

Theorem B. *Let $f: X \rightarrow Y$ in \mathcal{NH} . Then the following statements are equivalent:*

- (i) f P -localizes,
- (ii) $\pi_n f: \pi_n X \rightarrow \pi_n Y$ P -localizes for all $n \geq 1$, and
- (iii) $H_n f: H_n X \rightarrow H_n Y$ P -localizes for all $n \geq 1$.

2. Results. In this paper, we define the P -local dimension as follows: the P -local dimension of a space X is at most n (denoted by $\dim_P X \leq n$) provided that every map $f: A \rightarrow S_P^n$ of a closed subset A of X into a P -local n -dimensional sphere S_P^n admits a continuous extension over X . Recall that a space X is said to have cohomological dimension with respect to a coefficient group $G \leq n$, written $c\text{-dim}_G X \leq n$, provided, for every map $f: A \rightarrow K(G, n)$ of a closed subset A of X into an Eilenberg-MacLane space $K(G, n)$ of type (G, n) there is an extension to a map $F: X \rightarrow K(G, n)$. By the dimension of a space X (denoted by $\dim X$) we mean the *covering dimension* of X . \mathbf{Z} is the additive group of all integers and \mathbf{Q} is the additive group of all rational numbers. $\mathbf{Z}_{(P)}$ is the ring of integers localized at P , that is, the subring \mathbf{Q} consisting of rationals expressible as fractions k/l with $l \in P'$. We denote by \mathbf{Z}_p and \mathbf{Z}_{p^∞} the cyclic group of order p and the quasicyclic group of type \mathbf{Z}_{p^∞} , respectively.

First we shall see the following basic properties:

Proposition 1. *If $\dim_P X \leq n$, then $\dim_P X \leq n + 1$.*

Proposition 2. *Let X be a metrizable space. We have the following inequality:*

$$c\text{-dim}_{\mathbf{Z}_{(P)}} X \leq \dim_P X.$$

In particular, if X is finite dimensional or ANR's, the equality holds.

Proposition 3. *Let X be a metrizable space. We have the following equality:*

$$c\text{-dim}_{\mathbf{Q}} X = \dim_{\mathbf{Q}} X.$$

Theorem 4. *Let $P_1 \subseteq P_2 \subseteq \mathcal{P}$. Then we have the following inequality:*

$$\dim_{P_1} X \leq \dim_{P_2} X.$$

We shall illustrate by the following example the essential differences between the theory of cohomological dimension and the theory of P -local dimension.

Example 5. There exists a compactum X such that $\dim_2 X \geq 3$ and $c\text{-dim}_{\mathbf{Z}_{(2)}} X \leq 2$.

Remark 6. The above example follows that $\text{Tor}(\mathbf{Z}_2, \pi_q(S^2)) \neq *$ for infinitely many q [Se]. Note that Serre's proof used Poincaré series and methods of analytic number theory.

The remainder is devoted to developing the main results.

A finite collection P_1, \dots, P_s of subsets of \mathcal{P} is called a *partition of \mathcal{P}* if $P_1 \cup \dots \cup P_s = \mathcal{P}$ (we do not assume that P_i are pairwise disjoint).

Theorem 7. *Let X be a compactum. Then the following conditions are equivalent:*

- (1) $\dim X < \infty$,
- (2) for some partition P_1, \dots, P_s of \mathcal{P} , $\max\{\dim_{P_i} X : i = 1, \dots, s\} < \infty$,
- (3) for any partition P_1, \dots, P_s of \mathcal{P} , $\max\{\dim_{P_i} X : i = 1, \dots, s\} < \infty$.

Remark 8. There exists an infinite dimensional compactum X such that for any partition P_1, \dots, P_s of \mathcal{P} , $\max\{c\text{-dim}_{\mathbf{Z}_{(P_i)}} X : i = 1, \dots, s\} < \infty$.

Remark 9. Let $\mathcal{P} = \{p_1, p_2, \dots\}$. There is an infinite dimensional compactum Y such that $\dim_{p_i} Y = i$ for $i \in \mathbf{N}$.

Remark 10. By Theorem 7 and an argument of cohomological dimension, we have

$$\dim X = \sup\{\dim_{P_i} X : i = 1, \dots, s\}$$

for any partition P_1, \dots, P_s of \mathcal{P} .

We note that the above does not hold for non-compact spaces [Dr-R-S].

Corollary 11. *Let X be a compactum and P_1, \dots, P_s be a partition of \mathcal{P} . Then if $\dim_{P_i} X = c\text{-dim}_{\mathbf{Z}_{(P_i)}} X$ for $i \in \{1, \dots, s\}$, $\dim X = c\text{-dim}_{\mathbf{Z}} X$.*

Remark 12. There is a compactum such that $\dim X = c\text{-dim}_{\mathbf{Z}} X = \infty$, $\dim_2 X \geq 3$ and $c\text{-dim}_{\mathbf{Z}_{(2)}} X \leq 2$.

We can get the following Menger-Urysohn's type sum formula for the local dimension.

Theorem 13. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$\dim_P X \leq \dim_P A + \dim_P B + 1.$$

Corollary 14. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$c\text{-dim}_{\mathbf{Q}} X \leq c\text{-dim}_{\mathbf{Q}} A + c\text{-dim}_{\mathbf{Q}} B + 1.$$

In particular, if X is finite dimensional, then the inequality with respect to $\mathbf{Z}_{(p)}$ holds.

Next, we shall develop the relation between localization and cohomological dimension.

Theorem 15. *Let G be an abelian group. We have the following equality:*

$$c\text{-dim}_G X = \sup\{c\text{-dim}_{G_p} X : p \in \mathcal{P}\}.$$

Corollary 16. *Let X be a finite dimensional compactum and K be a simply connected CW-complex. The following are equivalent:*

- (1) $K \in AE(X)$,
- (2) $K_p \in AE(X)$ for each prime $p \in \mathcal{P}$.

REFERENCES

- [Bo] B. F. Bockstein, *Homological invariants of topological spaces, I*, (English translation in Amer.Math. Soc.Transl. 11:3 (1950)), Trudy Moskov. Mat. Obshch. **5** (1956), 3–80. (Russian)
- [Dr₁] A. N. Dranishnikov, *Homological dimension theory*, Russian Math. Surveys **43:4** (1988), 11–62.
- [Dr-R-S] ———, D. Repovš and E. Schepin, *Dimension of products with continua*, preprint.
- [D-T] A.Dold and R.Thom, *Quasifaserungen und Unendliche Symmetrische Produkte*, Annals of Math. **67** (1958), 239–281. (German)
- [D₁] J.Dydak, *Cohomological dimension and metrizable spaces*, Trans. of the Amer.Math.Soc. **337** (1993), 219–234.
- [D₂] ———, *Cohomological dimension and metrizable spaces II*, preprint.
- [D-W] ——— and J.J.Walsh, *Infinite dimensional compacta having cohomological dimension two: an application of the Sullivan conjecture*, Topology **32** (1993), 93–104.
- [H-R-M] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematical Studied 15, Amsterdam, 1975.
- [H-W] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, Princeton, 1941.
- [Ko] Y.Kodama, *Appendix to K. Nagami, Dimension theory*, Academic Press, New York, 1970.
- [Ku] W. I. Kuzminov, *Homological dimension theory*, Russian Math. Surveys **23** (1968), 1–45.
- [R] J. Roitberg, *Note on nilpotent spaces and localization*, Math. Z. **137** (1974), 67–74.
- [Se] J. P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comm. Math. Helv. **27** (1953), 198–232.
- [Sp] E.Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
- [Su] D.Sullivan, *Geometric Topology, Part I: Localization, Periodicity, and Galois Symmetry*, M.I.T. Press, 1970.
- [Wa] J.J.Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Lecture Notes in Math. **870**, 1981, pp. 105–118.
- [Wh] George W.Whitehead, *Elements of homotopy theory*, Springer-Verlag, 1978.
- [Z] A. Zabrodsky, *On phantom maps and a theorem of H. Miller*, Israel J. Math. **58** (1987), 129–143.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA-SHI, IBARAKI, 305, JAPAN
E-mail address: yokoi@sakura.cc.tsukuba.ac.jp